Notes on Fibre Bundles

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Abstract

The concept of fibre bundle is introduced. It constitutes for example the proper framework for presenting general relativity and gauge theory. Here the focus is put on gauge theory, and thus on principal bundles.

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1 Introduction

A *manifold* generalizes the concept of curves and surfaces, the definition being given without reference to an embedding in R^n. It is a topological space which is homeomorphic to R^n locally, but not necessarily so globally. A *fibre bundle* is a topological space which looks like the direct product of two topological spaces , locally, but in general not so globally.

2 Definition

We denote a differentiable **fibre bundle** with *total space* E, (typical) *fibre* F over the *base space* M, *structure group* G and *projection* π, by (E, π, M, F, G). E, M, F are differentiable manifolds. G is a Lie group acting on F on the left. $\pi : E \to M$ is a surjection; for any $p \in M$, the fibre at p is identified with $\pi^{-1}(p) \equiv F_p \cong F$. A *local trivialization* is a diffeomorphism

$$\phi_i : U_i \times F \to \pi^{-1}(U_i)$$

such that $\pi\phi_i(p, f) = p$, where $\{U_i\}$ is an open covering of M. On $U_i \cap U_j \neq \emptyset$, ϕ_i and ϕ_j are related by the *transition function* $t_{ij} : U_i \cap U_j \to G$ defined as

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)$$

in which the left action of G on F has been used.

The transition functions need to satisfy the consistency conditions: (i) $t_{ii} = id_G$, (ii) $t_{ij} = t_{ij}^{-1}$, (iii) $t_{ij}t_{jk} = t_{ik}$. In case all the transition functions can be taken to be identity maps, the fibre bundle is denoted **trivial**.

A **local section** is a smooth map $s : M \supset U \to E$ which satisfies $\pi s = id_M$, a (global) section being the equivalent object in which U can be taken as the entire base space M.

In case the fibre is a vector space, $F \cong R^k$, the fibre bundle is called **vector bundle**. Examples are the *tangent*, *cotangent*, and *line* bundles. In this case the structure group is a sub-group of $GL(k)$, $dimE \equiv k$ being the fibre dimension.

3 Principal bundles

We will now focus on **principal bundle**, which is a fibre bundle, denoted $P(M, G)$, in which the fibre is identical with the structure group, $F \equiv G$.

In addition to the left action of G on F previously referred to, a *right action* of G on P may be now introduced. It is defined as

$$ua \equiv \phi_i(p, g_i a)$$

where $a \in G$ and $\pi^{-1}(p) \ni u = \phi_i(p, g_i)$, and the definition is actually independent of local trivialization.

The right action of G is *transitive* on the fibre $\pi^{-1}(p) = \{ua | a \in G\}$ (i.e., the orbits under the action are the fibres over each $p \in M$) and *free* ($ua = u$ for some $u \in P \Rightarrow a = e \equiv id_G$).

Given a local section s_i it can be introduced the corresponding **canonical local trivialization** $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ defined by

$$\phi_i^{-1}(u) = (p, g_u)$$

where $u \in \pi^{-1}(p)$, $p \in U_i$ and $g_u \in G$ is the unique element of G such that $u = s_i(p)g_u$. In this local trivialization the section s_i is expressed as $s_i(p) = \phi_i(p, e)$ and related to s_j by $s_i(p) = s_j(p)t_{ij}(p)$. Conversely, a trivialization ϕ_i defines a local section $s_i : U_i \to \pi^{-1}(U_i)$ through the equation

$$s_i = \phi^{-1} \; o \; \bar{id}$$

where $\bar{id} : U_i \to U_i \times G$ is given by $x \mapsto (x, e)$.

Given a principal fibre bundle $P(M, G)$ and a manifold F on which G acts on the left, it can be constructed an **associated fibre bundle** (E, π, M, G, F, P) as the equivalence class

$$E \sim (P \times F)/G$$

that results from the action of G on $P \times F$ given by $(u, f) \mapsto (ug, g^{-1}f)$.

It is shown that a *principal* bundle is trivial iff it admits a global section. A corollary of this result states that a *vector* bundle is trivial iff its *associated principal bundle* admits a global section.

4 Connections on principal bundles

There are several equivalent ways of defining connections on a principal fibre bundle. For convenience, here we consider one based on the separation of the tangent space T_uP into *vertical* V_uP and *horizontal* H_uP subspaces,

$$T_uP = V_uP \oplus H_uP$$

.

Let $u \in P(M, G)$ and $G_p = \pi^{-1}(p)$ be the fibre at $p = \pi(u) \in M$. The **vertical subspace** $V_uP \subset T_uP$ is tangent to G_p at u. It thus corresponds to the set of equivalent classes of paths on $P(M, G)$ contained in G_p, which paths can be generated by the right action of G on G_p. In particular the elements of VP can be generated by the *induced vector fields* by the elements of the Lie algebra $\mathcal{L}(G)$ of G through its right action on P — $A^{\#} \in VP$ correspond at u to the derivative of the curve in G_p given by

$$R_{\exp(tA)}u = u\exp(tA)$$

where A is an element of the Lie algebra of G, $\exp : \mathcal{L}(G) \to G$ is the exponential map (from the algebra to the group), and $\exp(tA)$ is the *one-parameter subgroup* of G generated by $A \in \mathcal{L}\,(G)$. There is thus a vector space isomorphism between $\mathcal{L}(G)$ and V_uP,

$$\mathcal{L}(G) \cong V_uP$$

The *horizontal subspace* H_uP is the complement of V_uP in T_uP. It is uniquely specified when a connection is defined in P. [1]

A **connection** on P is a unique separation $T_uP = V_uP \oplus H_uP$ such that (i) a smooth vector field X on P is separated into smooth vectors $X^V \in V_uP$ and $X^H \in H_uP$ as $X = X^V + X^H$, and (ii) $H_{ug}P = R_{g*}H_uP$.

This separation can be accomplished by the **connection one-form** or **Ehresmann connection** ω. It is defined as a Lie algebra-valued one-form

$$\omega \in \mathcal{L}(\mathcal{G}) \otimes \Omega^1(P)$$

which projects T_uP onto the $V_uP \sim \mathcal{L}(G)$, namely: (i) $\omega(A^{\#}) = A$ with $A \in \mathcal{L}(G)$, (ii) $R_g^*w = Ad_{g^{-1}}w \iff R_g^*w_{ug}(X) \equiv w_{ug}(R_{g*}X) = g^{-1}w_u(X)g$ for $X \in T_uP$. It is shown that the definition of ω satisfies indeed the two defining properties of a connection previously stated.

The **horizontal subspace** is now properly defined as the kernel of ω

$$H_uP \equiv \{X \in T_uP | w(X) = 0\}$$

5 Local connection forms on the base manifold

Let $\sigma_i : U_i \to \pi^{-1}(U_i)$ be a local section, with $\{U_i\}$ an open covering of M.

Given an Ehresmann connection $\omega \in \mathcal{L}(G) \otimes \Omega^1(P)$, a Lie algebra-valued one-form $\mathcal{A}_i$ can be formed on U_i just by performing the *pullback* of ω with σ_i

$$\mathcal{A}_i \equiv \sigma_i^* w \quad \in \mathcal{L}(G) \otimes \Omega^1(U_i)$$

Conversely, given a local Lie algebra-valued one-form $\mathcal{A}_i$ on U_i, it can always be reconstructed a connection one-form ω whose pullback by σ_i is $\mathcal{A}_i$. Indeed, define

$$w_i \equiv g_i^{-1}\pi^*\mathcal{A}_i g_i + g_i^{-1}d_P g_i$$

[1] indeed, it introduces implicitly a metric that specifies a systematic procedure for finding the complement space to V_uP

where d_P is the exterior derivative on P and g_i is such that $u = \sigma_i(p)g_i$, i.e. it is the fibre element associated to u in the *canonical local trivialization.* It can be shown that $\sigma_i^* w_i \equiv \mathcal{A}_i$, and furthermore that ω_i satisfies the axioms of a connection one-form.

For the uniqueness of ω on P, i.e. assuring that $\omega_i = \omega_j$ on $U_i \cap U_j$, the local forms $\mathcal{A}_i$ are required to satisfy the following compatibility condition

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1} dt_{ij}$$

$\mathcal{A}_i$ is denoted the **gauge potential**.

Many connection one-forms can be defined (globally) over the principal bundle. However all share the same global information about the bundle. It's ω, or the transition functions, and not the *individual* local pieces $\mathcal{A}_i$, that contain the global information of P.

5.1 Gauge transformation

Let $P(M, G)$ be a principal bundle, σ_1 and σ_2 two local sections over a chart U of the base space M, such that $\sigma_2(p) = \sigma_1(p)g(p)$. It is seen from the compatibility condition above that then the corresponding gauge potentials $\mathcal{A}_1$ and $\mathcal{A}_2$ are related as

$$\mathcal{A}_2 = g^{-1}\mathcal{A}_1 g + g^{-1}dg$$

which in local coordinates is written as

$$\mathcal{A}_{2\mu}(p) \; = \; g^{-1}(p) \; \mathcal{A}_{1\mu}(p) \; g(p) \; + \; g^{-1}(p) \; \partial_\mu \; g(p)$$

which is commonly referred to as the **gauge transformation** for $\mathcal{A}$.

6 Parallel transport

The **horizontal lift** of a curve $\gamma : [0,1] \to M$ on the base space M with respect to a point $u_0 \in \pi^{-1}(\gamma(0))$ is defined as the *unique* curve $\tilde{\gamma} : [0,1] \to P$ on the principal bundle $P(M, G)$ for which $\tilde{\gamma}(0) = u_0$, $\pi\tilde{\gamma} = \gamma$, and the tangent vector to $\tilde{\gamma}(t)$ always belongs to $H_{\gamma(t)}P$, *i.e.* is horizontal. Given a local section σ_i over U_i, the horizontal lift of γ can be expressed as $\tilde{\gamma}(t) = \sigma_i(\gamma(t)) \; g_i(\gamma(t))$ with

$$g_i(\gamma(t)) \; = \; P \; \exp \; (-\int_0^t \mathcal{A}_{i\mu} \; \frac{dx^\mu}{dt} \; dt \;) \; = \; P \; \exp \; (-\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}_{i\mu}(\gamma(t)) \; dx^\mu \;)$$

where P is a *path-ordering* operator along γ.

The unique horizontal lift $\tilde{\gamma}$ of γ through u_0 defines unique point $u_1 = \tilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$; u_1 is called the **parallel transport** of u_0 along the curve γ. In local form, it is given by

$$u_1 \;=\; \sigma_i(\gamma(1))\; P\;\exp\;(-\int_0^t \mathcal{A}_{i\mu}\frac{d\,x^\mu(\gamma(t))}{dt}\;dt\;)$$

This introduces a map $\Gamma(\gamma) : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(t))$ defined as $u_0 \mapsto u_1$.

If $\tilde{\gamma}$ and $\tilde{\gamma}'$ are two horizontal lifts of γ such that $\tilde{\gamma}'(0) \;=\; \tilde{\gamma}(0)\; g$, then $\tilde{\gamma}'(t) \;=\; \tilde{\gamma}(t)\; g,\;\; \forall\; t$. This result ensures in particular that $\Gamma(\gamma)$ and the right action R_g commute

$$R_g\;\Gamma(\gamma) \;=\; \Gamma(\gamma)\; R_g$$

7 Holonomy

Consider a loop $\gamma : [0,1] \to M$, $\gamma(0) = \gamma(1) = p$ on the base space of a principal bundle $P(M,G)$. Then the parallel transport of a point $u \in \pi^{-1}(\gamma(0)) \equiv \pi^{-1}(\gamma(1))$ along the loop does not in general coincide with u; *i.e.*, in general, $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$.

A loop γ defines thus a transformation $\tau_\gamma : \pi^{-1}(p) \to \pi^{-1}(p)$. τ_γ commutes with the right action $\tau_\gamma(ug) = \tau_\gamma(u)g$.

Consider the set of loops $C_p(M)$ at a point $p \in M$, $C_p(M) \equiv \{\; \gamma : [0,1] \to M \mid \gamma(0) = \gamma(1) = p\; \}$. The **holonomy group** at a point $u \in \pi^{-1}(p)$ is

$$\Phi_u \;\equiv\; \{\; g \in G \mid \tau_\gamma(u) = ug, \gamma \in C_p(M)\; \}$$

The group structure follows from the homotopy group structure in M.

The **restricted holonomy group** at u

$$\Phi_u^0 \;\equiv\; \{\; g \in G \mid \tau_\gamma(u) = ug, \gamma \in C_p^0(M)\; \}$$

where $C_p^0(M) \equiv \{\gamma \in C_p(M) \mid \gamma \textit{ is homotopic to constant loop at } p\}$, is obtained by considering only trivial loops in M.

8 Curvature

8.1 Covariant derivative in principal bundle

This corresponds to a generalization of the the exterior derivatives of (real-valued) r-forms to the case of vector valued r-forms.

Let $\Omega^r(P) \ni \phi \equiv \sum_{\alpha=1}^k \phi^\alpha \otimes e_\alpha : TP \otimes \ldots \otimes TP \to V$, with V a vector space of dimension k, with basis $\{e_\alpha\}_{\alpha=1}^k$, $\phi^\alpha \in \Omega^r(P)$ and $X_1 \ldots X_{r+1} \in T_uP$.

The **covariant derivative** of ϕ is defined as

$$D\phi(X_1,\ldots,X_{r+1}) \equiv d_P\phi(X_1^H,\ldots,X_{r+1}^H)$$

where $d_P\phi \equiv d_P\phi^\alpha \otimes e_\alpha$, and X_i^H is the horizontal component of $X_i = X_i^V + X_i^H$ according to the decomposition of $T_uP = V_uP \oplus H_UP$ provided by a connection one-form on the principal fibre bundle $P(M, G)$.

8.2 The curvature two-form

The **curvature two-form** Ω is the covariant derivative of the connection one-form ω,

$$\Omega \equiv D\omega \ \in \ \Omega^2(P) \otimes \mathcal{L}(\mathcal{G})$$

Ω and ω satisfy *Cartan's structure equation*

$$\Omega = d_P\omega + \omega \wedge \omega$$

which, for $X, Y \in T_uP$, can be written as

$$\Omega(X, Y) = d_P\omega(X, Y) + [\omega(X), \omega(Y)]$$

(note that $[\omega, \omega]$ denotes both the wedge product of the forms and the Lie bracket of the vectors involved).

The exterior differentiation of the curvature gives

$$d_P\Omega \ = \ d_P(d_P\omega + \omega \wedge \omega) \ = \ d_P\omega \wedge \omega - \omega \wedge d_P\omega$$

therefore, for the covariant derivative,

$$D\Omega(X, Y, Z) \ = \ d_P\Omega(X^H, Y^H, Z^H) \ = \ 0$$

which is the **Bianchi identity**, $D\ \Omega \ = \ 0$. In local form [2]

$$\begin{aligned} d\mathcal{F} \equiv d\ \sigma^*\Omega = \sigma^*\ d_P\Omega = \sigma^*(d_P\omega \wedge \omega - \omega \wedge d_P\omega) &= d\sigma^*\omega \wedge \omega - \omega \wedge d\sigma^*\omega \\ &= d\mathcal{A} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{A} = \mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F} = -[\mathcal{A}, \mathcal{F}] \end{aligned}$$

$$\Leftrightarrow \qquad \mathcal{D}\ \mathcal{F} \ \equiv \ d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$$

Given horizontal vectors $X, Y \in H_uP$, the vertical component $[X, Y]^V$ of the Lie bracket of $X = X^H$ and $Y = Y^H$ is given [3] by $\Omega(X, Y)$,

$$\Omega(X, Y) \ = \ d_P\omega(X, Y) \ = \ -\omega([X, Y]) \ = \ A$$

where A is an element of $\mathcal{L}(G)$ such that $[X, Y] = A^\#$ ($\#$ denotes the isomorphism $\mathcal{L}(G) \cong V_uP$).

[2] we anticipate here this definition that is given next

[3] employing the general formula $d\omega(X, Y) = X[w(Y)] - Y[w(X)] - w([X, Y])$ for a differential form ω and vectors X, Y

As noted before, the horizontal lift $\tilde{\gamma}$ of a path γ fails to close, and the failure is given by the holonomy. Also, the curvature measures this discrepancy, namely the distance between the initial and the final point which belong to the same fibre. Indeed, the holonomy group is expressed in terms of the curvature, as the following result shows:

The *Ambrose-Singer theorem* states that, for the case M is compact,

$$\mathcal{L}(G) \supset span(\{\Omega_u(X,Y) \mid X,Y \in H_uP\}) \quad \cong \quad \mathcal{L}(\Phi_{u_0})$$

where u_0 and u belong to the same horizontal lifting.

8.3 The local form of the curvature

Given a local section $\sigma : \; M \supset U \to P$, the local form $\mathcal{F}$ of the curvature Ω is given by

$$\mathcal{F} \equiv \sigma^* \Omega$$

This definition is similar to the case of the connection $\mathcal{A} = \sigma^*\omega$, in terms of which $\mathcal{F}$ is expressed [4] as

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

or equivalently, applied to vectors $X, Y \in TM$,

$$\mathcal{F}(X,Y) = d\mathcal{A}(X,Y) + [\mathcal{A}(X), \mathcal{A}(Y)]$$

In a chart $U \subset M$ with local coordinates $\{x^\mu\}$, expressing $\mathcal{A}$ and $\mathcal{F}$ as

$$\mathcal{A} \;=\; \mathcal{A}_\mu \; dx^\mu \qquad \mathcal{F} \;=\; \frac{1}{2}\, \mathcal{F}_{\mu\nu} \; dx^\mu \wedge dx^\nu$$

allows one to write

$$\mathcal{F}_{\mu\nu} \;=\; \partial_\mu \, \mathcal{A}_\nu - \partial_\nu \, \mathcal{A}_\mu \;+\; [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

The components $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ are $\mathcal{L}(G)$-valued functions, and therefore can be expanded as

$$\mathcal{A}_\mu = \mathcal{A}^a_\mu T_a \qquad \mathcal{F}_{\mu\nu} = \mathcal{F}^a_{\mu\nu} T_a$$

in terms of a basis of $\mathcal{L}(G)$, with $[T_a, T_b] = f_{ab}{}^c T_c$, from which

$$\mathcal{F}^a_{\mu\nu} \;=\; \partial_\mu \, \mathcal{A}^a_\nu - \partial_\nu \, \mathcal{A}^a_\mu \;+\; f_{bc}{}^a \; \mathcal{A}^b_\mu \; \mathcal{A}^c_\nu$$

[4] employing Cartan's structure equation

$\mathcal{F}$ is denoted the *Yang-Mills* **field strength**.

On the overlapping region of two charts $U_i \cap U_j \neq \emptyset$, the compatibility condition for $\mathcal{A}$ gives

$$\mathcal{F}_i \ = \ Ad_{t_{ij}^{-1}} \ \mathcal{F}_j \ = \ t_{ij}^{-1} \ \mathcal{F}_j \ t_{ij}$$

with t_{ij} the transition function on $U_i \cap U_j$, which is the form of the *gauge transformation* for $\mathcal{F}$.